Hale School

Mathematics Specialist

Term 2 2017

Test 4 - Integration

SECTION ONE

Name: Hemy Sophia

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Instructions:

- **SECTION ONE: CAS calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of SECTION ONE: 30 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 **(9 marks)**

Determine the following integrals:

a) $\int \frac{\sin^2(2x)+\cos^2(2x)}{\cos^2(4x)}\, dx$ (3 marks)

$= \int \frac{1}{\cos^2 4x}\, dx$

$= \frac{1}{4}\tan 4x + C$

✓ Uses identity $\sin^2 x + \cos^2 x = 1$

✓ Recognises $\int \frac{1}{\cos^2 x}\, dx = \tan x$

✓ Correct constant

b) $\int \frac{\cos^3 x}{3}\, dx$ (3 marks)

$= \int \frac{1}{3}\cos x \cos^2 x \, dx$

$= \int \frac{1}{3}\cos x\,(1-\sin^2 x)\, dx$

$= \int \frac{1}{3}\cos x - \frac{1}{3}\sin^2 x \cos x \, dx$

$= \frac{1}{3}\sin x - \frac{1}{9}\sin^3 x + C$

✓ uses identity $\sin^2 x + \cos^2 x = 1$

✓ Expands correctly

✓ Integrates correctly

c) $\int_0^1 \frac{e^{ax}}{1+e^{ax}}\, dx$ where a is a constant. (3 marks)

$\left[\frac{1}{a}\ln|1+e^{ax}|\right]_0^1$

$= \frac{1}{a}\ln(1+e^a) - \frac{1}{a}\ln(1+e^0)$

$= \frac{1}{a}\ln(1+e^a) - \frac{1}{a}\ln 2$

$= \frac{1}{a}\ln\left(\frac{1+e^a}{2}\right)$

✓ Recognises $\ln x$ in integral

✓ Substitutes values correctly

✓ fully simplifies.

Question 2 **(~~7~~ 8 marks)**

Using the substitution $\cancel{u}\,x = 2\cos u$ determine the following definite integral:

$$\int_0^1 \sqrt{4-x^2}\,dx$$

Let $x = 2\cos u$ When $x = 1$ $u = \frac{\pi}{3}$

$\frac{dx}{du} = -2\sin u$ $x = 0$ $u = \frac{\pi}{2}$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4-(2\cos u)^2}\; -2\sin u\, du$$

✓ differentiates substitution

✓ changes bounds.

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4\sin^2 u}\; -2\sin u\, du$$

✓ replaces variables in one step

✓ uses $\sin^2 x + \cos^2 x = 1$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -4\sin^2 u\, du$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -4\left(\frac{1}{2} - \frac{1}{2}\cos 2u\right) du$$

✓ uses $2\sin^2 u = 1 - \cos 2u$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -2 + 2\cos 2u\, du$$

$$= \Big[-2u + \sin 2u\Big]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$$

✓ Integrates correctly

$$= \left(-2\left(\frac{\pi}{3}\right) + \sin 2\left(\frac{\pi}{3}\right)\right) - \left(-2\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right)\right)$$

✓ Substitute values

$$= -\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - (-\pi)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

✓ Simplifies correctly

Question 3 **(8 marks)**

Determine the following integral:

$$\int \frac{3x^2+13x-16}{3x^2+2x-8}\,dx$$

$$= \int \frac{3x^2+2x-8+11x-8}{3x^2+2x-8}\,dx$$ ✓ Divides to get 1

$$= \int 1 + \frac{11x-8}{3x^2+2x-8}\,dx$$ ✓ Correct remainder $11x-8$

$$\frac{11x-8}{3x^2+2x-8} = \frac{A}{(3x-4)} + \frac{B}{(x+2)}$$ ✓ Factorises quadratic

$$\therefore 11x-8 = A(x+2) + B(3x-4)$$ ✓ Equates Numerators

when $x=-2$ $\quad 11(-2)-8 = B(3(-2)-4)$

$$-30 = -10B$$

$$B = 3$$ ✓ Solves for B

when $x=\frac{4}{3}$ $\quad \frac{44}{3} - 8 = \left(\frac{4}{3}+2\right)A$

$$\frac{\frac{20}{3}}{\frac{10}{3}} = A$$ ✓ Solves for A

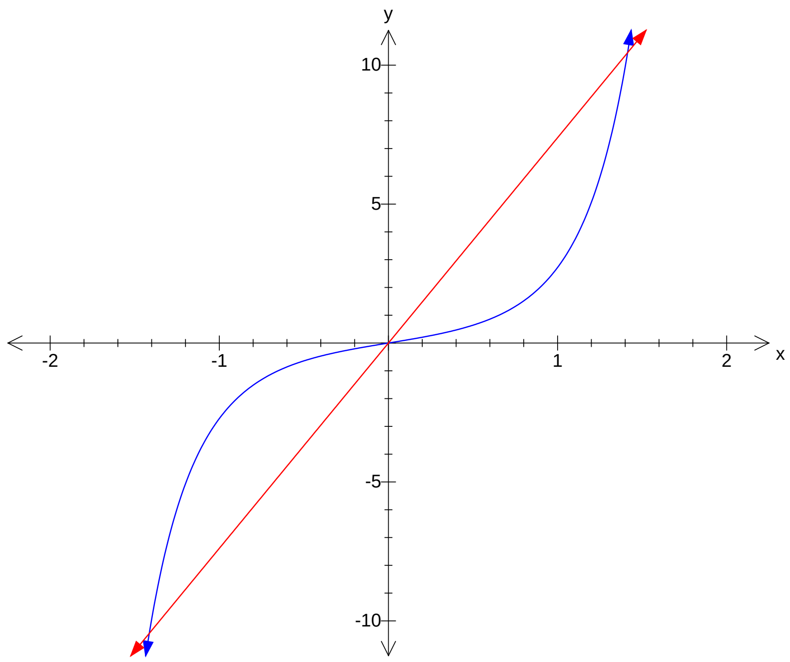
$$A = 2.$$

$$= \int 1 + \frac{2}{3x-4} + \frac{3}{x+2}\,dx$$ ✓ Establishes Integral

$$= x + 2\ln|3x-4| + 3\ln|x+2| + c$$ ✓ Integrates correctly.

Question 4 **(5 marks)**

The graphs defined by $y = xe^{x^2}$ *and* $y = e^2 x$ are shown below. Calculate the **exact** area enclosed between the two curves.



Intersection $xe^{x^2} = e^2 x$

$\therefore \quad e^{x^2} = e^2$

$x^2 = 2$

$x = \pm\sqrt{2}, 0.$

✓ determines intersection

$\therefore$ Area $= 2\int_0^{\sqrt{2}} e^2 x - xe^{x^2} \, dx$

✓ establishes correct integral

✓ multiplys by two or finds second section

$= 2\left[\frac{e^2x^2}{2} - \frac{e^{x^2}}{2}\right]_0^{\sqrt{2}}$

✓ integrates correctly

$= \left(e^2\sqrt{2}^2 - e^{(\sqrt{2})^2}\right) - \left(-\frac{e^0}{2}\right)$

$= \left(2e^2 - e^2\right) + \frac{1}{2}$

$= e^2 + \frac{1}{2}$

✓ evaluates correctly

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SECTION TWO

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Name: ______________________________

Instructions:

- **SECTION TWO: CAS calculators are allowed**
- **External notes are not allowed**
- **Duration of SECTION TWO: 15 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 5 **(7 marks)**

A sphere of radius 10 cm is formed by rotating the semi-circle $y = \sqrt{100 - x^2}$ about the x-axis.

a) Using volume of revolution determine the volume of this sphere. (2 marks)

$$\int_{-10}^{10} \pi (100 - x^2) \, dx$$ ✓ shows use of v.o.r.

$$= \frac{4000\pi}{3} \text{ cm}^3$$ ✓ evaluates correctly.

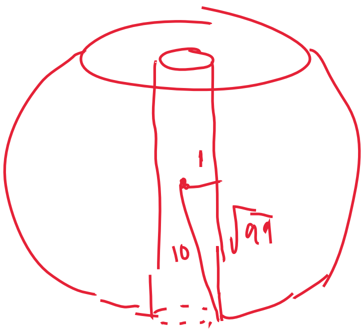
(4188.79)

b) The sphere of radius 10 cm has a cap of height 2 cm removed from the top. Find the volume of the spherical cap. (2 marks)

$$\int_{8}^{10} \pi (100 - x^2) \, dx$$ ✓ establishes integral

$$= \frac{112\pi}{3} \text{ cm}^3$$ (117.29) ✓ evaluate correctly

c) The remaining portion of the sphere has a cylindrical hole of radius 1 cm bored symmetrically from the top of the cut sphere, directly through the centre to the other side. Find the volume of the sphere remaining. (3 marks)



$$V = \pi \int_{-\sqrt{99}}^{8} (100 - y^2) - 1^2 \, dy$$

$$= \frac{1864\pi}{3} + 198\sqrt{11}\,\pi \text{ cm}^3$$

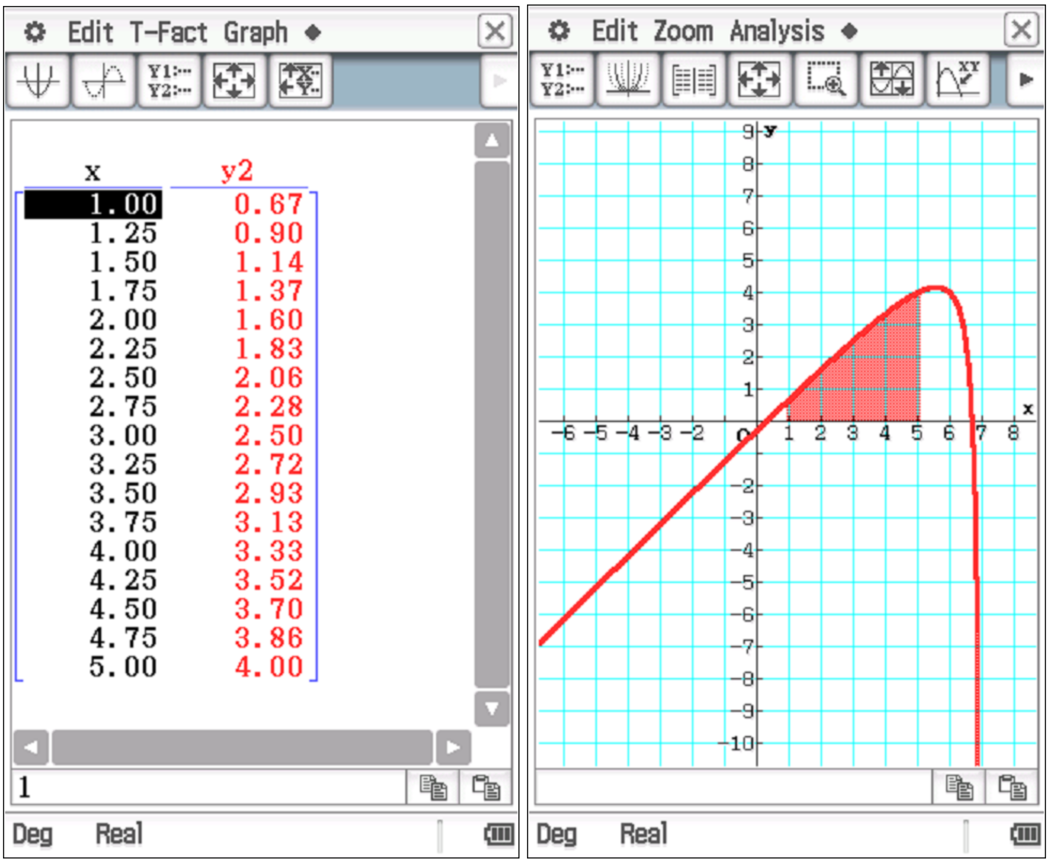
(4015.03)

✓ correct bounds

✓ rotates around y axis

✓ correct answer

Question 6 **(6 marks)**



x	y2
1.00	0.67
1.25	0.90
1.50	1.14
1.75	1.37
2.00	1.60
2.25	1.83
2.50	2.06
2.75	2.28
3.00	2.50
3.25	2.72
3.50	2.93
3.75	3.13
4.00	3.33
4.25	3.52
4.50	3.70
4.75	3.86
5.00	4.00

The shaded region R in the diagram above shows the region trapped between the curve $y=f(x)$, the x-axis and the lines $x=1$ and $x=5$. The accompanying table shows the value of the function $f(x)$ (indicated as y2 from the calculator input) for the various values of x.

a) Estimate the area of R using the trapezium rule with 8 strips. (3 marks)

$A \approx 0.5 \times \left\{ \frac{f(1)+f(1.5)}{2} + \ldots + \frac{f(4.5)+f(5)}{2} \right\}$ ✓ states rules

$\approx \frac{0.5}{2} \times \{0.67 + 2(1.14+1.60+2.06+2.50+2.93+3.33+3.70) + 4\}$ ✓ uses correct values

$= 9.7975$

✓ correct result to at least 2 d.p. accuracy.

b) Estimate the area of R using Simpson's rule with 4 strips. (3 marks)

$A = \frac{1}{3} \times \{f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)\}$ ✓ states simpson's rule

$= \frac{1}{3} \times (0.67 + 4.00 + 2(2.5) + 4(1.60 + 3.33)\}$ ✓ uses correct values

$= 9.7967$

✓ correct result to at least 2 d.p. accuracy.

END OF TEST